NEW SETS OF KINK BEARING HAMILTONIANS

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ABSTRACT

Given a kink bearing Hamiltonian, Isospectral Hamiltonian approach is used in generating new sets of Hamiltonains which also admit kink solutions. We use Sine-Gordon model as a example and explicitly work out the new sets of potentials and the solutions.

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Nonlinear field theory models in (1+1) dimensions, with Lagrangian density $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \ - V(\phi)$, for which the equations of motion admit finite energy, finite width solutions have been well studied [1]. Sine-Gordon model and ϕ^4 models are two popular models used for modelling some physical systems. While the solutions of ϕ^4 are called kinks/lumps, the solutions of Sine-Gordon model have the extra property that they retain their identity after collisions, and they are called solitons. In order to model a variety of physical systems, various attempts are made to enlarge the class of nonlinear field theoretic models. Parametrically modified Sine-Gordon model was one of such earlier attempts made by Remoissenet and Peyrard [2]. In this model the potential is $V(\phi, r)$ whose shape can be varied continuosly as a function of r, $-1 < r < 1$ and has the Sine-Gordon (SG) potential for $r = 0$ as a special case. The implicit kink solutions for this model and their rest masses are calculated. For $r \neq 0$ the model is not completely integrable. Kink-antikink interactions are studied for this model and was shown that the structure is similar to that observed in $K\bar{K}$ interactions for the ϕ^4 model for a range of r [3].

In this article we present a prescription to construct a family of potentials which admit kink solutions. The method is based on Isospectral Hamiltonian approach which enables us to generate two sets of potentials from a given potential. In one case we can give explicit kink solution whereas implicit solutions are obtained in the other case. A partial result of this approach using ϕ^4 model was presented earlier by one of us [4].

Once the field theoretic model admits kink type solutions, the stability of the kink is ensured by the occurence of the zero energy ground state of the stability equation when small oscillations around the kink are considered [1]. Considering the stability equation as a one dimensional Schrodinger-like equation for a particle in a potential $V(x)$ we can construct a isospectral partner for it. Then, as we explain below, following the work of Christ and Lee [5] we shall construct the kink solution and the potential which admits the solution from the partner stability equation.

Let the Lagrangian density of a single hermitian scalar field ϕ in 1+1 dimensions be

given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \ - V(\phi) \tag{1}$$

In order to have kink solutions the potential $V(\phi)$ is assumed to have atleast two degenerate absolute minima. The time independent field equations reads

$$\frac{d^2\phi}{dx^2} - \frac{dV}{d\phi} = 0 \tag{2}$$

which can also be written as

$$\frac{1}{2}(\frac{d\phi}{dx})^2 = V(\phi) \tag{3}$$

on taking squareroot of equn (3) and integrating

$$\int [\sqrt{2V(\phi)}]^{-1} \ d\phi \ = \ x + x_0 \tag{4}$$

This relation gives ϕ in terms of x an implicit solution. For the cases where ϕ can be expressed in terms of x the explicit solutions can be obtained. The stability of the kink solution is ensured by the occurence of the zero energy ground state of the corresponding stability equation. Expanding the ϕ around kink solution as $\phi \ = \phi_k \ + \psi$ one gets

$$[-\frac{d^2}{dx^2} + V''(\phi_k)]\psi_n = E_n \psi_n \tag{5}$$

It is straight forward to see that

$$E_0 = 0, \quad \text{and} \quad \psi_0 = \frac{d\phi_k}{dx} \tag{6}$$

satisfies the equation (5). Infact the relation between the ψ_0 and ϕ_k is the result of translational invariance of the Lagrangian, field equation and the solution [1]. The equations (3) and (6) play an important role in our analysis.

We can consider the stability equation as a 1-dimensional Schordinger like equation for a particle in 1-dimensional potential. The knowledge of the ground state energy and the ground state wavefunction enables to factorise the equation. Once the factorisation is done as $H = A^+A^- + \epsilon_0$ the energy spectrum of the H and its partner $H_1 = A^-A^+ + \epsilon_0$ are

related by supersymmetry as $E_1^{(n)} = E^{n+1}, (n = 0, 1, 2, 3...)$ where $E,\ E_1$ are the energy spectrum of $H,\ H_1$ respectively [6]. The operators $A^{\pm}$ are given by $\frac{1}{\sqrt{2}}(\pm\frac{\partial}{\partial x} + \alpha(x))$, where $\alpha(x) = \frac{d}{dx}\log\psi_0$, ψ_0 is the ground state wavefunction of H. Given a parent Hamiltonian H with N energy levels, one can construct the 'daughter' Hamiltonian H_1 with $N-1$ levels and the procedure can be iterated. At the H_1 level the question can be asked, following Mielnek [7] whether the factorisation of H_1 is unique or not. Consider $H_1 = B^- B^+ + \epsilon_0$ is another factorisation where $B^{\pm} = \frac{1}{\sqrt{2}}(\pm\frac{\partial}{\partial x} + \beta(x))$. An obivous particular solution is $\beta(x) = \alpha(x)$. Let the general solution be $\beta(x) = \alpha(x) + \chi(x)$ This yields

$$\chi^2(x) + 2\chi(x)\alpha(x) - \chi'(x) = 0$$

This is a Riccatti equation whose solution is

$$\chi(x) = \psi_0^2\ (C - \int \psi_0^2 dx)^{-1}$$

where $\alpha(x) = \frac{d}{dx}\log\psi_0$, and C is a integration constant chosen such that χ is non-singular. Hence we have another factorisation i.e. $H_1 = A^- A^+ + \epsilon_0 = B^- B^+ + \epsilon_0$. At this stage the above equation is little new to offer. However if we construct $B^+ B^-$ it is no longer H but a new Hamiltonian H^N

$$H^N = H_1 + \frac{\partial}{\partial x}\beta(x)$$

We call H^N an isospectral partner of H in the sense that it has same energy spectrum as that of H. The ground state wave of H^N is given by

$$\psi_0^N = \psi_0 (C - \int \psi_0^2 dx)^{-1} \tag{7}$$

In a sense nonuniqueness of factorising H has led us to construct one more parent Hamiltonian H^N in addition to the original Hamiltonian [4].

Now that the partner stability equation has been constructed, following the ref. 5 (using equns. 3 and 6) the kink solution and the corresponding field theory admitting the solution can be obtained, but a slight subtlety enables us to generate two sets of Hamiltonians as we explain here. In what follows we shall keep the discussion general

as well work out the Sine-Gordon model as a specific example. Using the equn. (6) in equn.(7) and integration gives

$$\phi_k^N = \int \psi_0 (C - \int \psi_0^2 dx)^{-1} dx \tag{8}$$

again to remind C is an integration constant earlier chosen such that χ is nonsingular. ϕ_k^N is the new kink solution and the potential is given by the equation (3).

For the specific case of SG potential with $m = 1$, $V(\phi) = (1 - \cos\phi)$ and the soliton solution is $4\tan^{-1}\exp\ x$. The zero energy translational mode reads $\psi_0 = sechx$ therefore

$$\psi_0^N = \frac{sechx}{(C - tanhx)} \quad , \quad |C| > \ 1$$

integrating this equation gives

$$\phi_k^N(x) = \frac{1}{\sqrt{C^2 - 1}} \tan^{-1}(sinh(x - x_0)), \qquad x_0 = \tanh^{-1}(\frac{1}{C}) \tag{9}$$

The kink solution asymptotically ($x \to \pm\infty$) takes the values $\pm\frac{\pi}{\sqrt{(C^2-1)}}$. On using the equn.(3) the potential takes the form

$$V(\phi) = \frac{1}{2(C^2 - 1)} \cos^2(\sqrt{C^2 - 1}\ \phi) \tag{10}$$

It has to be mentioned here that, for the Sine-Gordon example the new potential and the kink solution are just rescaled and translated versions of the original potential and kink solution respectively. In the case of ϕ^4 model this procedure does lead to different result. But the algebraic complexity of expressing x in terms of ϕ and thus $V(\phi)$ ($\frac{1}{2}(d\phi/dx)^2$) as a function of ϕ hindered the further study of the potential [4].

In the second case, the starting point would be to use the equns. (3,6,7) together instead of integrating equn. (7) in conjunction with equn.(6). We have

$$\frac{1}{2}(\frac{d\phi}{dx})^2 = V(\phi) \tag{3$'$}$$

$$\psi_0 = \frac{d\phi}{dx} \tag{6$'$}$$

$$\psi_0^N = \psi_0 (C - \int \psi_0^2 dx)^{-1} \tag{7'}$$

Substituting(6′) in (7′) and squaring the resulting equation it is straight forward to see that

$$V(\phi^N) = \frac{V(\phi)}{(C - \int \sqrt{2V(\phi)}\ d\phi)^2} \tag{11}$$

In the case of SG model $V(\phi) = (1 - \cos\phi)$ and the $V^N(\phi)$ reads

$$V^N(\phi) = \frac{1}{16} \frac{(1 - \cos\phi)}{(\frac{C}{4} + \cos\phi/2)^2} \quad , \qquad |C| > 4 \tag{12}$$

The kink solution in this case is an implicit one and has the form

$$2\ \tan^{\frac{C}{4}-1}(\frac{\phi}{4})\ \sin^2(\frac{\phi}{4}) = \exp\frac{(x + x_0)}{4} \tag{13}$$

The asymptotic behaviour is as follows. As $x \to \infty, \quad \phi = 2\pi$ as $x \to -\infty, \quad \phi = 0$. The potential in this case has 4π periodicity, therefore two types of kink solutions exist. One was described above and the second solution $\phi_2(x)$ takes the values $(2\pi, 4\pi)$ as $x \to \pm\infty$ and the solution is given by $\phi_2(x) = 4\pi - \phi_1(-x)$. The representative profiles of the potential and the kink solution are plotted for three values of C (see Figs.1 and 2). It is interesting to note that while minima of the potential are pegged at the same point the maxima are dependent on the parameter C.

Two remarks are in order. (1). While in the first case the kink solution is obtained first using equn.(6) and the potential is derived using equn.(3), whereas in the second case the potential is obtained first and then the solution is derived using the standard integration, resulting two different sets of kink bearing Hamiltonians. (2). The potential for the SG model in the second case is different when compared to the case considered in ref.2 $V(\phi)$ in their case is given by

$$V(\phi) = (1 - r)^2 \frac{(1 - \cos\phi)}{(1 + r^2 + 2r\cos\phi)} \quad , \quad |r| < 1 \tag{14}$$

SG case can be obtained in their case for the value $r = 0$.

In conclusion, using the Isospectral Hamiltonian approach, we constructed two new sets of Hamiltonians from a given Hamiltonian which admits kink solutions. SG model has

been worked out as a specific example. We feel the second way of generating potentials is useful in the light of Peyrard and Campbell work on kink-antikink interactions, earlier referred to in this article. A related problem in these cases is to examine if the present kinks are of soliton or quasi-soliton type.

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FIGURE CAPTIONS

Fig. 1. The potential $V^N(\phi)$ (equn. 12) for (a) C=5 (b) C=7 and (c) C=9

Fig. 2. The kink profile (equn. 13) for (a) C=5 (b) C=7 and (c) C=9.